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TOPOLOGICAL ENTROPY AND THE PSEUDO-ORBIT TRACING PROPERTY

by

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ABSTRACT

We show an inequality of the topological entropies between semiconjugate dynamical systems on compact Hausdorff spaces and apply this inequality to the bundle map on a fiber bundle whose total space, the base space and the structure group are compact Hausdorff spaces. A new method of calculating the topological entropy of a continuous map from a compact Hausdorff space to itself is given. The topological entropy $h(f)$ of the expansive homeomorphism f with the pseudo-orbit tracing property from a compact metric space to itself satisfies the equality

$$h(f) = \limsup_{n \rightarrow \infty} (1/n) \cdot \log N_n(f),$$

where $N_n(f)$ is the number of fixed points of f^n .

0. INTRODUCTION

Let X be a compact space. We denote by $OC(X)$ the set of all the open coverings of X . For a continuous map $f: X \rightarrow X$ the topological entropy $h(f)$ is defined as follows. For $\alpha \in OC(X)$ and $n \in \mathbb{N}$, we write

$$\alpha_f^n = \{ \bigcap_{j=1}^{n-1} f^{-j} A_j ; A_j \in \alpha, 0 \leq j < n \} (\in OC(X)), \quad (1)$$

For any subset $K \subset X$ and $\alpha \in OC(X)$, we write

$$N_K(\alpha) = \min \{ \# \beta ; \beta \subset \alpha, K \subset \bigcup_{B \in \beta} B \}, \quad (2)$$

where $\# \beta$ denotes the cardinality of β .

Then the topological entropy $h(f, K)$ of f with respect to K is

defined by

$$h(f, K) = \sup_{\alpha \in OC(X)} \limsup_{n \rightarrow \infty} (1/n) \cdot \log N_K(\alpha_f^n) . \quad (3)$$

Of course, $h(f, K)$ coincides with the topological entropy $h(f)$ defined by R.L.Adler, A.G.Konheim and M.H.McAndrew¹⁾.

THEOREM 1.1. Let X be a compact space and Y a compact Hausdorff space. Let $f: X \rightarrow X$, $g: Y \rightarrow Y$ and $\pi: X \rightarrow Y$ be continuous maps satisfying $\pi \circ f = g \circ \pi$ and $\pi(X) = Y$. Then the following inequality holds.

$$h(f) \leq h(g) + \sup_{y \in Y} h(f, \pi^{-1}(y)) \quad (4)$$

This has been shown by R.Bowen³⁾ in the case that X and Y are compact metric spaces.

THEOREM 1.2. Let $\pi: E \rightarrow X$ be a projection of a fiber bundle with the total space E and the base space X . Assume that E , X and the structure group are compact Hausdorff spaces and that $f: E \rightarrow E$ is a bundle map whose base map is $f': X \rightarrow X$, then

$$h(f) = h(f') . \quad (5)$$

We say here (X, f) is a cascade if X is a compact Hausdorff space and $f: X \rightarrow X$ is a continuous map.

In §2 we show that the topological entropy of a cascade can be calculated by using finite closed coverings.

THEOREM 3.1. Let (X, d) be a compact metric space and $f: X \rightarrow X$ an expansive homeomorphism (Resp. a positively expansive continuous map) with the pseudo-orbit tracing property (Resp. the positive pseudo-orbit tracing property) . Then it follows that

$$h(f) = \limsup_{n \rightarrow \infty} (1/n) \cdot \log N_n(f) \quad (6)$$

where

$$N_n(f) = \#\{ x \in X ; f^n(x) = x \} \quad (n \in \mathbb{N}) . \quad (7)$$

This has been shown by R. Bowen²⁾ for a homeomorphism on a compact metric space with hyperbolic canonical coordinates, and K. Hiraide⁵⁾ has given this result by showing that any expansive homeomorphism with the pseudo-orbit tracing property on a compact metric space has Markov partitions of arbitrary small diameter.

1. QUOTIENTS

In this section we sketch proofs of Theorem 1.1. and Theorem 1.2. .

Sketch of a proof of Theorem 1.1.

Take $\alpha \in OC(X)$ and $n \in \mathbb{N}$ arbitrarily. Then for each $y \in Y$ and a subset $\beta \subset \alpha_f^n$ such that $\pi^{-1}(y) \subset \bigcup_{B \in \beta} B$, there exists an open subset U_y of Y such that $y \in U_y$ and $\pi^{-1}(U_y) \subset \bigcup_{B \in \beta} B$. Because Y is a compact Hausdorff space and X is compact. Then $\gamma = \{U_y; y \in Y\}$ is an element of $OC(Y)$. Take $C \in \gamma_g^{ln}$ for each $l \in \mathbb{N}$. Then we can see the following inequality.

$$N_{\pi^{-1}(C)}(\alpha_f^{ln}) \leq \left[\sup_{y \in Y} N_{\pi^{-1}(y)}(\alpha_f^n) \right]^l \quad (7)$$

Since

$$N_X(\pi^{-1}(\gamma_g^{ln})) \leq N_Y(\gamma_g^{ln}) \quad (9)$$

where

$$\pi^{-1}(\gamma_g^{ln}) = \{\pi^{-1}(C); C \in \gamma_g^{ln}\}, \quad (10)$$

we see

$$N_X(\alpha_f^{ln}) \leq \left[\sup_{y \in Y} N_{\pi^{-1}(y)}(\alpha_f^n) \right]^l \cdot N_Y(\gamma_g^{ln}). \quad (11)$$

Because $\lim_{n \rightarrow \infty} (1/n) \log N_X(\alpha_f^n)$ exists (see R.L. Adler et al.¹⁾), we see

$$h(f, X, \alpha) \leq \sup (1/n) \log N_{\pi^{-1}(y)}(\alpha_f^n) + h(g, Y, \gamma). \quad (12)$$

Since $n \in \mathbb{N}$ and $\alpha \in OC(X)$ are arbitrary, we have the desired inequality.

Sketch of a proof of Theorem 1.2.

We have to show,

$$\sup_{x \in X} h(f, \pi^{-1}(x)) = 0. \quad (13)$$

Take $\alpha \in OC(E)$ and $x \in X$. Assume we can find an open covering β of $\pi^{-1}(x)$ such that β refines $\{A \cap \pi^{-1}(x) ; A \in \alpha_f^n\}$, i.e. for any $B \in \beta$ there exists $A \in \alpha_f^n$ such that $B \subset A$, for all $n \in \mathbb{N}$, so that $h(f, \pi^{-1}(x), \alpha) = 0$. Since α is arbitrary, we have $h(f, \pi^{-1}(x)) = 0$. But the equicontinuity of the action of the structure group implies the existence of such an β for each $\alpha \in OC(X)$ and $x \in X$.

2. A METHOD OF CALCULATING TOPOLOGICAL ENTROPY

Let $s \in \mathbb{N}$ be a positive integer and $A = (A_{ij})$ an (s, s) -matrix whose entries are 0 or 1. Set $S = \{1, \dots, s\}$, then for each $n \in \mathbb{N}$ we denote the set of all the sequences (a_0, \dots, a_{n-1}) S^n of length n which satisfies $A_{a_j a_{j+1}} = 1$ for all j ($0 \leq j < n$) by $M_n(A)$.

Let (X, f) be a cascade.

DEFINITION 2.1. The pair (α, A) of indexed finite closed covering $\alpha = \{F_1, \dots, F_s\}$ of X and (s, s) -matrix $A = (A_{ij})$ whose entries are all 0 or 1 is said to be a CM-pair for (X, f) , if

$$X = \bigcup_{a \in M_n(A)} \bigcap_{j=0}^{n-1} f^{-1} F_{a_j} \quad \text{where } a = (a_0, \dots, a_{n-1}). \quad (13)$$

Let (α, A) be a CM-pair for (X, f) . For $n \in \mathbb{N}$, a subset $P \subset M_n(A)$ is said to be separated if for any distinct elements $p, p' \in P$ there exists j ($0 \leq j < n$) such that $F_{p_j} \cap F_{p'_j} = \emptyset$ where $F_i \in \alpha$ ($0 \leq i \leq s$, $s = \#\alpha$) and $p = (p_0, \dots, p_{n-1})$ etc.. And for $n \in \mathbb{N}$ and a subset $K \subset X$, a subset $P \subset M_n(A)$ is said to be attached to K if $K \cap \bigcap_{j=0}^{n-1} f^{-1} F_{p_j} \neq \emptyset$ for all $(p_0, \dots, p_{n-1}) \in P$ where $F_i \in \alpha$ ($0 \leq i \leq s$, $s = \#\alpha$).

We set,

$$S_n(f, K, (\alpha, A)) = \max \{ \#P; P \subset M_n(A), \\ P \text{ is both separated and attached to } K \}, \quad (14)$$

and

$$\overline{S}_f(K, (\alpha, A)) = \limsup_{n \rightarrow \infty} (1/n) \cdot \log S_n(f, K, (\alpha, A)). \quad (15)$$

Then the topological entropy $h(f, K)$ with respect to K is given as follows.

PROPOSITION 2.1. Let Γ be a family of CM-pairs for (X, f) . Assume that for any $\alpha_0 \in OC(X)$ there exists $(\alpha, A) \in \Gamma$ such that α refines α_0 , then

$$h(f, K) = \sup \overline{S}_f(K, (\alpha, A)) \quad ((\alpha, A) \in \Gamma) \quad (16)$$

Proof. Let (α, A) be an arbitrary CM-pair for (X, f) . For each $x \in X$, set $O(x) = \bigcap \{F^c; F \in \alpha, x \notin F\}$ where F^c is the complement of a subset $F \subset X$. Then $\beta = \{O(x); x \in X\} \in OC(X)$. Let $F \in \alpha$ and $x \in X$ satisfy $F \cap O(x) \neq \emptyset$. Then $x \in F$ from the definition. In particular, if $F, F' \in \alpha$ are such that $F \cap F' = \emptyset$, then for each $x \in X$, $F \cap O(x) = \emptyset$ or $F' \cap O(x) = \emptyset$. From this one sees that for any separated set $P \subset M_n(A)$ ($n \in \mathbb{N}$), each $\bigcap_{j=0}^{n-1} f^{-j} O(x_j) \in \beta_f^n$ ($x_j \in X, 0 \leq j < n$) can intersect at most one element of $\{\bigcap_{j=0}^{n-1} f^{-j} F_{p_j}; (p_0, \dots, p_{n-1}) \in P\}$. This implies the following inequality,

$$N_K(\beta_f^n) \geq S_n(f, K, (\alpha, A)) \quad \text{for all } n \in \mathbb{N}. \quad (17)$$

And this implies that $h(f, K)$ is larger than the right hand side of the equation (16).

On the other hand, for $\beta_0 \in OC(X)$ and $B \in \beta_0$ set

$$U(B) = \bigcup \{B'; B' \in \beta_0, B \cap B' \neq \emptyset\}. \quad (18)$$

Fix an arbitrary $\alpha_0 \in OC(X)$, then there exists $\beta_0 \in OC(X)$ such

that $\gamma = \{U(B); B \in \beta_0\}$, refines α_0 . From the assumption there exists $(\alpha, A) \in \Gamma$ such that α refines β_0 . Fix $n \in \mathbb{N}$ and let $P \subset M_n(A)$ be a maximal separated set attached to K . For each $x \in K$, because of the equation (13) for (α, A) , there exists $a \in M_n(A)$ such that $x \in \bigcap_{j=0}^{n-1} f^{-j} F_{a_j}$ where $F_i \in \alpha$ ($1 \leq i \leq s$, $s = \#\alpha$) and $a = (a_0, \dots, a_{n-1})$. Then there exist $p = (p_0, \dots, p_{n-1}) \in P$ such that $F_{a_j} \cap F_{p_j} \neq \emptyset$ for all j ($0 \leq j < n$), so that taking B_i ($F_i \subset B_i \in \beta$) for each i ($0 \leq i \leq S$) we can find

$$x \in \bigcap_{j=0}^{n-1} f^{-j} F_{a_j} \subset \bigcap_{j=0}^{n-1} f^{-j} B_{a_j} \subset \bigcap_{j=0}^{n-1} f^{-j} U(B_{p_j}) \in \gamma_f^n. \quad (19)$$

This implies

$$N_K(\gamma_f^n) \leq S_n(f, X, (\alpha, A)). \quad (20)$$

Since α_0 is refined by γ , we have

$$h(f, K, \alpha_0) \leq \sup \overline{S}_f(K, (\alpha, A)) \quad ((\alpha, A) \in \Gamma). \quad (21)$$

Since $\alpha_0 \in OC(X)$ is arbitrary we are done.

3. PERIODIC POINTS

Let $f: X \rightarrow X$ be a homeomorphism on a compact metric space

(X, d) . Let $\delta > 0$. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points of X is a δ -pseudo-orbit (δ -p.o.) if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let

$\varepsilon > 0$. A point $x \in X$ ε -traces a δ -p.o. $\{x_i\}_{i \in \mathbb{Z}}$ if

$d(f^i(x), x_{i+1}) \leq \varepsilon$ for all $i \in \mathbb{Z}$. A homeomorphism f from a compact metric space (X, d) to itself has the pseudo-orbit tracing property

(P.O.T.P.) if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every

δ -p.o. $\{x_i\}_{i \in \mathbb{Z}}$ ($x_i \in X$, $i \in \mathbb{Z}$) is ε -traced by some $x \in X$ depending on the δ -p.o. $\{x_i\}_{i \in \mathbb{Z}}$.

A homeomorphism $f: X \rightarrow X$ on a compact metric space (X, d) is expansive if there exists $\varepsilon > 0$ such that for all distinct elements $x, y \in X$ there exists $n \in \mathbb{Z}$ such that $d(f^n(x), f^n(y)) > \varepsilon$.

THEOREM 3.1. Let (X, d) be a metric space. And let $f: X \rightarrow X$ be an expansive homeomorphism (Resp. a positively expansive continuous map) with P.O.T.P. (Resp. positive P.O.T.P.), then it follows that

$$h(f) = \limsup_{n \rightarrow \infty} (1/n) \cdot \log N_n(f) \quad (22)$$

where

$$N_n(f) = \# \{x \in X; f^n(x) = x\} \quad (n \in \mathbb{N}). \quad (23)$$

Proof. We omit a proof.

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